Ergodic Optimal Harvesting Strategies for a Predator-Prey System in Fluctuating Environments.

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(joint work with Dang Nguyen)

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Outline

- Harvesting is one of the central issues in bio-economics. It has been widely recognized that it may not be a good idea to consider only maximizing short-term benefits focusing purely on harvesting. Although over-harvesting in a short period may maximize the short-term economic benefits, it breaks the balance between harvesting and its ecological implications.
- Simple minded policies may lead to detrimental after effect. It is crucially important to pay attention not to render exceedingly harmful decision to the environment.
- For some optimal harvesting models with finite-time yield or discounted yield, see, Alvarez (1998, 2006), Alvarez et al (2016), Lungu and Oksendal (2001), among others.

- In contrast, ecologists and bio-economists emphasize the importance of sustainable harvest in both biological conservation and long-term economic benefits. Anderson and Seijo (2010) and Clark (2010) introduced the concept of *maximum sustainable yield*, which is the largest yield (or catch) that can be taken from a species' stock over an infinite horizon.
- They indicated that it is more reasonable to maximize the yield in such a way that a species is sustainable and not in danger leading to extinction of the species.
- Inspired by the idea of using maximum sustainable yield, we pay special attentions to sustainability, biodiversity, biological conservation, and long-term economic benefits, and consider long-term horizon optimal strategies. In lieu of discounted profit, we examine objective functions that are of long-run average per unit time type.

- The equation for a Lotka-Volterra predator-prey system perturbed by white noise is

$$\begin{cases} dX(t) = X(t)\left[a_1 - b_1 X(t) - c_1 Y(t)\right]dt + X(t)\sigma_1 dW_1(t) \\ dY(t) = Y(t)\left[-a_2 - b_2 Y(t) + c_2 X(t)\right]dt + Y(t)\sigma_2 dW_2(t). \end{cases} \tag{1.1}$$

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- Putting an harvesting effort $u(t)$ at time t on the predator, the equation becomes

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- $u(t)$ takes value in an interval $[0, M]$. Thus, the amount of harvested biomass in a short period of time Δt is $Y(t)u(t)\Delta t$.

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- The time-average harvested value over an interval $[0, T]$ is $\frac{1}{T}\int_0^T \Phi\Big(Y(t)u(t)\Big)dt$.
- Our goal is to

$$\text{maximize } \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \Phi\Big(Y(t)u(t)\Big)dt \text{ a.s.} \tag{1.3}$$

HJB Equation of Ergodic Control Problem for Diffusion Processes

- Consider the controlled diffusion on $\mathbb{R}^d$

$$dX(t) = b(X(t), u(t))dt + \sigma(X(t))dW(t) \tag{2.1}$$

  where $u(\cdot) : \mathbb{R}_+ \mapsto U$, a compact metric space.
- $b(x,u)$, $\sigma(x)$ are Lipschitz in x uniformly in u and have linear growth rates.
- $\sigma\sigma^\top$ is uniformly elliptic.

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- $b(x,u)$, $\sigma(x)$ are Lipschitz in x uniformly in u and have linear growth rates.
- $\sigma\sigma^\top$ is uniformly elliptic.
- Our goal is to minimize a.s. over all admissible policies the functional

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T c(X(t), u(t))dt. \tag{2.2}$$

- Suppose that c is continuous in (x,u) and bounded below.

- We broaden the class of admissible controls to “relaxed controls”. Let $\mathscr{V} = \mathscr{P}(U)$ be the space of probability measures endowed with the Prohorov topology.
- A relaxed control is a measurable function $\mathbb{R}_+ \mapsto \mathscr{V}$.
- A precise control (an ordinary control) is a relaxed control with Dirac measure values.

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- Define $\overline{b}, \overline{c} : \mathbb{R}^d \times \mathscr{V} \mapsto \mathbb{R}^d$ by

$$\overline{b}(\cdot, v) = \int_U b(\cdot, u) v(du); \qquad \overline{c}(\cdot, v) = \int_U c(\cdot, u) v(du);$$

- With a relaxed control $v(t)$, the equation becomes

$$dX(t) = \overline{b}(X(t), v(t))dt + \sigma(X(t))dW(t) \tag{2.3}$$

and our goal is to minimize

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- Henceforth, "control" means *relaxed control*. "non-randomized" control means *precise control* (ordinary control).

- $\tilde{v}(t)$ is called a Markov control (feedback control) if $\tilde{v}(t) = v(X(t))$ for a measurable map $v : \mathbb{R}^d \mapsto \mathscr{V}$
- It can be proved that under a Markov control v,

$$dX(t) = \overline{b}(X(t), v(X(t))dt + \sigma(X(t))dW(t)$$

  admits a unique strong solution that is a strong Feller process; see Krylov "Controlled Diffusion Processes".

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- A Markov control v is called stable if the corresponding process $X(t)$ is positive recurrent. In this case, the process has a unique invariant probability measure, denoted by $\eta_v \in \mathscr{P}(\mathbb{R}^d)$.
- Denote by Π, Π_{SM}, Π_{SMD} the sets of admissible controls, stable Markov controls, stable non-randomized Markov controls respectively.

- Under a stable Markov control, by the ergodicity we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \overline{c}(X(s), v(X(s))ds = \int_{\mathbb{R}^d} \overline{c}(x, v(x))\eta_v(dx) =: \rho_v, \text{ a.s.}$$

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- Suppose that $\rho^* := \inf_{v\in\Pi_{SM}}\{\rho_v\} < \infty$.
- Under some additional conditions, it can be shown that there is a Markov control v^* such that

$$\rho_{v^*} = \rho^* = \inf_{v\in\Pi_{SM}}\{\rho_v\}.$$

and

$$\limsup \frac{1}{T}\int_0^T c(u(t), X(t)dt \geq \rho^*, \text{ a.s.}$$

for any admissible control $u(\cdot)$.

- The condition imposed to penalize unstable behavior is

Assumption 2.1 (Near Monotonicty Condition)

$$\liminf_{\|x\|\to\infty}\left\{\inf_{u\in U} c(x,u)\right\} > \rho^*. \tag{2.5}$$

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$$\liminf_{\|x\|\to\infty}\left\{\inf_{u\in U} c(x,u)\right\} > \rho^*. \tag{2.5}$$

- Intuitively, (2.5) penalizes trajectories lying outside the set $\{\inf_{u\in U} c(x,u)\} \leq \rho^*$.
- It forces an optimal process to spend a nonvanishing fraction of time in a bounded compact set. It excludes "unstable" policy from being candidates for optimality.

Theorem 2.1

Suppose that Assumption 2.1 is satisfied and $c(x,u)$ is locally Lipschitz in x uniformly in u. Then there exists $V \in C^2(\mathbb{R}^d)$ such that

$$\min_{u \in U} [\mathscr{L}^u V(x) + c(x,u)] = \rho^*.$$

and the pair (V, ρ^*) is unique in the class of $(V, \rho) \in W^{2,p}_{loc}(\mathbb{R}^d) \times \mathbb{R}$ satisfying

$$\min_{u \in U} [\mathscr{L}^u V(x) + c(x,u)] = \rho$$

and

$$\rho \leq \rho^*, V(0) = 0, \inf_{\mathbb{R}^d} V > -\infty.$$

A Markov control v is optimal if and only if it satisfies $\min_{u \in U} [\mathscr{L}^u V(x) + c(x,u)] = \mathscr{L}^v V(x) + \bar{c}(x,v)$ *or equivalently*

$$\min_{u \in U} \left[\sum b^i(x,u) V_{x_i}(x) + \bar{c}(x,u)\right] = \sum \bar{b}^i(x,v) V_{x_i}(x) + \bar{c}(x,v)$$

- The condition we impose to penalize unstable behavior is as follows.

Assumption 2.2 (Uniform Stability Condition)

$$\liminf_{R\to\infty}\sup_{v\in\Pi}\int_{B_R^c\times U}\{1+|c(x,u)|\}\,\eta_v(dx)v(du)=0. \tag{2.6}$$

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$$\lim_{R\to\infty} \inf_{v\in\Pi} \sup \int_{B_R^c\times U} \{1+|c(x,u)|\}\,\eta_v(dx)v(du) = 0. \qquad (2.6)$$

Under Assumption (2.2), There exists an inf-compact function h such that

$$\lim_{\|x\|\to\infty} \frac{c(x,u)}{h(x)} = 0$$

and

$$\sup_{v\in\Pi} \int_{\mathbb{R}^n\times U} h(x)\eta_v(dx)v(du) < \infty$$

Theorem 2.2

Suppose that Assumption 2.2 is satisfied and $c(x,u)$ is locally Lipschitz in x uniformly in u. Then there exists $V \in C^2(\mathbb{R}^d)$ such that

$$\min_{u\in U}[\mathscr{L}^u V(x) + c(x,u)] = \rho^*$$

and

$$lim_{\|x\|\to\infty} \frac{V(x)}{h(x)} = 0 \tag{2.7}$$

and the pair (V,ρ^*) is unique in the class of $(V,\rho) \in W^{2,p}_{loc}(\mathbb{R}^d) \times \mathbb{R}$ satisfying (2.7) and $V(0) = 0$. A Markov control v is optimal if and only if it satisfies $\min_{u\in U}[\mathscr{L}^u V(x) + c(x,u)] = \mathscr{L}^v V(x) + \bar{c}(x,v)$ or equivalently

$$\min_{u\in U}\left[\sum b^i(x,u)V_{x_i}(x) + \bar{c}(x,u)\right] = \sum \bar{b}^i(x,v)V_{x_i}(x) + \bar{c}(x,v)$$

Optimal Harvesting Strategies for Single Population

- We consider a population whose density $\tilde{X}(t)$ at time $t \geq 0$, in the absence of harvesting, satisfies

$$d\tilde{X}(t) = \tilde{X}(t)(\mu - \kappa\tilde{X}(t))dt + \sigma\tilde{X}(t)dB(t), \ \tilde{X}(0) = x > 0,$$

where $(B(t))_{t\geq 0}$ is a standard one dimensional Brownian motion.

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where $(B(t))_{t\geq 0}$ is a standard one dimensional Brownian motion.

- Assume that the population is harvested at time $t \geq 0$ at the rate $h(t) \in U := [0, M]$,

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$$dX(t) = X(t)(\mu - \kappa X(t) - h(t))dt + \sigma X(t)dB(t), \ X(0) = x > 0. \tag{3.1}$$

- Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous, increasing function satisfying $\Phi(0) = 0$ and suppose Φ has a linear growth rate. Our aim is to find the optimal strategy $h(t)$ that almost surely maximizes

$$\liminf_{T\to\infty} \frac{1}{T} \int_0^T \Phi\Big(X(t)h(t)\Big)\,dt.$$

- For this controlled equation, neither near-monotonicity condition nor uniform stability condition are satisfied.

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- However, there are two boundary points 0 and ∞. the near-monotonicity condition is satisfied near 0, the uniform stability condition is satisfied near ∞.
- Since the two points are separate and the state space is one dimensional, we can obtain the desired result by assuming that

$$\mu - \frac{\sigma^2}{2} > 0.$$

Theorem 3.1

The HJB equation

$$\max_{u\in U}\left[\mathscr{L}_u V(x)+\Phi(xu)\right]=\rho \tag{3.2}$$

admits a classical solution $V^*\in C^2(\mathbb{R}_+)$ *satisfying* $V^*(1)=0$ *and* $\rho=\rho^*>0$*. The solution* V^* *of* (3.2) *has the following properties:*

a) *For any* $p\in(0,1)$

$$\lim_{x\to\infty}\frac{V^*(x)}{x^p}=0 \text{ and } \lim_{x\to 0}\frac{V^*(x)}{x^{-p}}=0 \tag{3.3}$$

b) *The function* V^* *is increasing.*

A Markov control v *is optimal if and only if it satisfies*

$$-\frac{dV^*}{dx}(x)xv(x)+\Phi(xv(x))=\max_{u\in U}\left(-\frac{dV^*}{dx}(x)xu+\Phi(xu)\right) \tag{3.4}$$

almost everywhere in $\mathbb{R}_+$.

Theorem 3.2

Assume that $\Phi(x) = x, x \in (0, \infty)$ *and that the population survives in the absence of harvesting, that is* $\mu - \frac{\sigma^2}{2} > 0$. *The optimal control* v *has the bang-bang form*

$$v(x) = \begin{cases} 0 & \text{if } 0 < x \le x^* \\ M & \text{if } x > x^* \end{cases} \tag{3.5}$$

for some $x^* \in (0, \infty)$. *Furthermore, we have the following upper bound for the optimal asymptotic yield*

$$\rho^* \le \frac{\mu^2}{4\kappa}. \tag{3.6}$$

Having the form (3.5) of the optimal harvesting strategy v in hand we next want to say something about the point x^* at which the harvesting strategy becomes nonzero. In order to achieve that we maximize over all the controls $w(\cdot,\eta)$ of bang-bang type

$$w(x;\eta) = \begin{cases} 0 & \text{if } 0 < x \leq \eta \\ M & \text{if } x > \eta, \end{cases} \tag{3.7}$$

and find the η which maximizes the asymptotic yield.

In this case the harvesting yield is

$$H(\eta) = \frac{\int_\eta^\infty yM \frac{1}{\sigma^2 y^2} \left(\frac{y}{\eta}\right)^{\frac{2(\mu-M)}{\sigma^2}} e^{-\frac{2\kappa}{\sigma^2}(y-\eta)} dy}{\int_0^\eta \frac{1}{\sigma^2 y^2} \left(\frac{y}{\eta}\right)^{\frac{2\mu}{\sigma^2}} e^{-\frac{2\kappa}{\sigma^2}(y-\eta)} dy + \int_\eta^\infty \frac{1}{\sigma^2 y^2} \left(\frac{y}{\eta}\right)^{\frac{2(\mu-M)}{\sigma^2}} e^{-\frac{2\kappa}{\sigma^2}(y-\eta)} dy} \tag{3.8}$$

Optimal Strategies of Havesting the Predator in a Predator-Prey Model

- $$\begin{cases} dX(t) = X(t)\big[a_1 - b_1X(t) - c_1Y(t)\big]dt + \sigma_1X(t)dW_1(t) \\ dY(t) = Y(t)\big[a_2 - u(t) - b_2Y(t) + c_2X(t)\big]dt + \sigma_2Y(t)dW_2(t), \end{cases} \tag{4.1}$$

  Our goal is to estalish the HJB equation and prove the existence and uniqueness of its solution.
- In the general setting for ergodic control of diffusion processes, the existence and uniqueness of solutions to the HJB equation is proved under either of two assumptions: near-monotonicity or uniformly stability of Markov controls.
- Our model satisfies neither of them although it partially satisfies a mixture of the two assumptions.

- We can show that

$$\lim_{t\to\infty} \mathbb{E}^{\nu}_{x,y}[X^2(t) + Y^2(t)] \le C$$

for some constant C that does not depend on x, y and ν. Thus, the controlled process is "uniformly stable" in the space $\mathbb{R}^2_+$, but NOT in $\mathbb{R}^{2,\circ}_+$.
- We have to handle the process on the boundary $x = 0$ and $y = 0$. When $X(t)$ is small, $Y(t)$ is decreasing to 0. That allows us to focus on the boundary $y = 0$. Near the boundary $y = 0$, the near-monotonicity condition is satisfied.
- Our model partially satisfies a mixture of the two assumptions.

Let $V_\alpha(s,x)$ be the optimal α-discounted yield, that is

$$V_\alpha(s,x) = \inf_{u \in \Pi_R} \mathbb{E}^u_{s,x} \int_0^\infty e^{-\alpha t} u(t) Y(t) dt, \ (s,x) \in \mathbb{R}^{2,\circ}_+.$$

Then $V_\alpha(s,x) \in C^2(\mathbb{R}^{2,\circ}_+) \cup C_b(\mathbb{R}^{2,\circ}_+)$ satisfies

$$\max_{u \in [0,M]} \left\{ \mathscr{L}^u V(x,s) + ux \right\} = \alpha V_\alpha(s,x) \tag{4.2}$$

Lemma 4.1 (A. Arapostathis, V. S. Borkar and M. K. Ghosh 2012)

Fix $(x_*, y_*) \in \mathbb{R}_+^{2,\circ}$. *For any sequence* $\alpha_n \downarrow 0$ *there exists a subsequence, which is still denoted by* $\{\alpha_n\}$, *and a function* $V \in C(\mathbb{R}^{2,\circ}$ *and a constant* ρ *such that as* $n \to \infty$, *we have*

$$\alpha_n V_{\alpha_n}(x_*, y_*) \to \rho \quad \text{and} \quad \overline{V}_{\alpha_n}(x,y) := V_{\alpha_n}(x,y) - V_{\alpha_n}(x_*, y_*) \to V(x,y) \tag{4.3}$$

uniformly in each compact subset of $\mathbb{R}^{2,\circ}$. *Moreover, we have*

$$\max_{u \in [0,M]} \{\mathscr{L}^u V(x,y) + \Phi(uy)\} = \rho \leq \rho^*, \ (x,y) \in \mathbb{R}_+^{2,\circ},$$

Conditions for Persistence

We assume that

$$a_1 - \frac{\sigma_1^2}{2} > 0$$

and

$$\lambda := -a_2 - \frac{\sigma_2^2}{2} + c_2 \frac{a_1 - \frac{\sigma_1^2}{2}}{b_1} > 0$$

Behaviors when $Y(t)$ is small

Without predator, the dynamics of the prey is

$$d\tilde{X}(t) = \tilde{X}(t)\left[a_1 - b_2\tilde{X}(t)\right]dt + \sigma_1\tilde{X}(t)dW_1(t)$$

Let $[\delta_1, \delta_1^{-1}]$, $\delta_1 < 1$ be a sufficiently large interval on $(0,\infty)$. Since $\lim_{T\to\infty 0}\frac{1}{T}\int_0^T \tilde{X}(t)dt = \frac{a_1 - \sigma_1^2/2}{b_2}$, there exists $T > 0$ such that

$$\frac{1}{T}\int_0^T \mathbb{E}_x\left(-a_2 - \frac{\sigma_2^2}{2} + \tilde{X}(t)\right)dt \geq \frac{3\lambda}{4}$$

for any $x \in [\delta_1, \delta_1^{-1}]$

Behaviors when $Y(t)$ is small

$$
\begin{aligned}
&\mathbb{E}^{v}_{x,y}\frac{\ln Y(T)-\ln Y(0)}{T}\\
&=-\left(a_2+\frac{\sigma_2^2}{2}\right)+\frac{1}{T}\mathbb{E}^{v}_{x,y}\int_0^T(c_2X(t)dt+v(X(t),Y(t))-b_2Y(t)dt)\\
&>\frac{\lambda}{2}-\frac{1}{T}\int_0^T v(X(t),Y(t))dt
\end{aligned}
$$

Intuiton

We should not harvest (or at least not harvest much) a species when its density is very small.
This tells us that an optimal strategy v^* should satisfies that $v^*(x, y)$ is small when y is small.

Lemma 4.2

There exists $\delta_2 > 0$ such that if v is a Markov control such that $\frac{1}{T}\mathbb{E}^{v}_{x,y}\int_0^T v(X(t),Y(t))dt > \frac{\lambda}{4}$ for some $(x,y) \in [\delta_1, \delta_1^{-1}] \times (0, \delta_2]$ then we can construct a Markov control $\tilde{v}$ such that

$$\frac{1}{T}\mathbb{E}^{\tilde{v}}_{x,y}\int_0^T v(X(t),Y(t))dt < \frac{\lambda}{4} \text{ for any } (x,y) \in [\delta_1, \delta_1^{-1}] \times (0, \delta_2]$$

and $\rho_v \le \rho_{\tilde{v}}$

The claim is also true for the value function of the α-discount problem when α is sufficiently small.

We can focus on the set of Markov control $v \in \tilde{\Pi}$ satisfying

$$\frac{1}{T}\mathbb{E}^{\tilde{v}}_{x,y}\int_0^T v(X(t),Y(t))dt < \frac{\lambda}{4} \text{ for any } (x,y) \in [\delta_1, \delta_1^{-1}] \times (0, \delta_2].$$

We denote that subset of controls by $\tilde{\Pi}$.
We can have

$$\mathbb{E}^{v}_{x,y}\frac{\ln Y(T) - \ln Y(0)}{T} \geq \frac{\lambda}{4}, (x,y) \in [\delta_1, \delta_1^{-1}] \times (0, \delta_2]$$

We analyze the log-Laplace transform to interchange the $\mathbb{E}$ and ln. to obtain $\mathbb{E}^{v}_{x,y}Y^{-\theta}(T) \leq \tilde{\gamma} y^{-\theta},\ \tilde{\gamma} \in (0,1)$

$$\mathbb{E}^{v}_{x,y} U^{\theta}(X(T), Y(T)) \le \gamma U^{\theta}(x,y), \gamma \in (0,1), \text{ for any } (x,y) \in \mathbb{R}^{2,\circ}_{+}$$

for $v \in \tilde{\Pi}$ where $\theta > 0$ is small enough and

$$U(x,y) = \frac{(1 + c_2 x + c_1 y)}{x^{0.5} y^{0.5}}$$

Since

$$|V(x,y)| \le \sup_{v \in \tilde{\Pi}} \mathbb{E}^{v}_{x,y} \int_0^{\tau_K} (\phi(v(X(t),Y(t))Y(t)) + \rho^*) dt + \sup_{(x,y) \in K} V$$

where $\tau_K = \inf\{t \ge 0 : (X(t), Y(t)) \in K\}$, we can show that

$$\lim_{x+y+\frac{1}{x}+\frac{1}{y} \to \infty} \frac{V(x,y)}{U^{\theta}(x,y)} = 0$$

Theorem 4.1

There is a unique pair of (V,ρ) *where* $V \in C^2(\mathbb{R}_+^{2,\circ}) \cap o(U)$ *and* $\rho \in \mathbb{R}$ *satisfying the equation*

$$\max_{u\in[0,M]} \left\{\mathscr{L}^u V(x,y) + \phi(uy)\right\} = \rho.$$

Moreover, we have $\rho = \rho^*$ *and* $v^* \in \Pi_{RM}$ *is an optimal control if and only if it is a measurable selector of from the maximizer:*

$$\max_{u\in[0,M]} \left\{\mathscr{L}^u V(x,y) + \phi(uy)\right\}.$$

In fact, if $\phi(uy) = uy$ *we can choose*

$$v^*(x,y) = \begin{cases} 0 & \text{if } \dfrac{\partial V(x,y)}{\partial y} \le 1 \\ M & \text{otherwise} \end{cases}$$

- Since $\max_{u\in[0,M]}\{\mathscr{L}^u V(x,s)+ux\}=\rho$ For any control $v\in\tilde{\Pi}$,

$$\begin{aligned}\mathbb{E}^v_{x,y}V(X(T),Y(T)-V(x,y)=&\mathbb{E}^v_{x,y}\int_0^T \mathscr{L}^v V(X(t),Y(t))dt\\ \leq&\mathbb{E}^v_{x,y}\int_0^T(\rho-\phi(v(X(t),Y(t))Y(t)))dt\end{aligned}$$

- Since $\frac{1}{T}\mathbb{E}^v_{x,y}V(X(T),Y(T))\to 0$ as $T\to 0$, we have that

$$\rho\geq\lim_{T\to\infty}\frac{1}{T}\mathbb{E}^v_{x,y}\int_0^T\phi(v(X(t),Y(t))Y(t)).$$

  Thus, $\rho\geq\rho^*$.

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  Thus, $\rho\ge\rho^*$.
- Let v^* be a control satisfying that

$$\mathscr{L}^v V(x,s)+\phi(vx)=\max_{u\in[0,M]}\left\{\mathscr{L}^u V(x,s)+\phi(ux)\right\},$$

  we can show that $\rho\le\rho^*$.

Havested Predator-Prey Model under Wideband Noise

Why Wideband Noise?

- It has been widely recognized that Brownian motion is only an idealized formulation or suitable limits of systems in the real world.
- To be more realistic, we would better assume that the environment is subject to disturbances characterized by a jump process with rapid jump rates. This jump process can be modeled by the so-called wideband noise.
- Motivated by the approach in Kushner and Ruggaldier (1978), we consider a Lotka-Volterra predator-prey model with wideband noise and harvesting in this paper.

The Model

- Denote by $X^\varepsilon(t)$ and $Y^\varepsilon(t)$ the sizes of the prey and the predator, respectively. The system of interest is of the form

$$\begin{cases} dX^\varepsilon(t) = & X^\varepsilon(t)\big[a_1 - b_1 X^\varepsilon(t) - c_1 Y^\varepsilon(t)\big]dt + \dfrac{1}{\varepsilon} X^\varepsilon(t) r_1(\xi^\varepsilon(t))dt \\ dY^\varepsilon(t) = & Y^\varepsilon(t)\big[-a_2 - h(Y^\varepsilon(t))u(t) - b_2 Y^\varepsilon(t) + c_2 X^\varepsilon(t)\big]dt \\ & + \dfrac{1}{\varepsilon} Y^\varepsilon(t) r_2(\xi^\varepsilon(t))dt, \end{cases} \tag{5.1}$$

- where ε is a small parameter, $\xi(t)$ is an ergodic, time-homogeneous, Markov-Feller process, and $\xi^\varepsilon(t) = \xi\left(\frac{t}{\varepsilon^2}\right)$, $a_i, b_i, c_i, i = 1, 2$ are positive constants, and $u(t)$ represents the harvesting effort at time t while $h(\cdot) : \mathbb{R}_+ \mapsto [0,1]$ indicates the effectiveness of harvesting, which is assumed to be dependent on the population of the predator.

The time-average harvested value over an interval $[0, T]$ is $\frac{1}{T}\int_0^T \Phi\Big(h(Y^\varepsilon(t)Y^\varepsilon(t))u(t)\Big)dt$. Our goal is to

$$\text{maximize } \liminf_{T\to\infty} \frac{1}{T}\int_0^T \Phi\Big(h(Y^\varepsilon(t))Y^\varepsilon(t)u(t)\Big)dt \text{ a.s.} \tag{5.2}$$

- Because of the complexity of the model, developing optimal strategies for the controlled system (5.1) and (5.2), are usually difficult. Nevertheless, one may wish to construct policies based on the limit system.
- A natural question arises: Can optimal or near-optimal harvesting strategies for the diffusion model be near optimal harvesting strategies for the wideband-width model when ε is sufficiently small?
- In a finite horizon, nearly optimal controls for systems under wideband noise perturbations were developed in the work of Kushner and Ruggaldier. For infinite horizon problems, under suitable conditions, the authors established that there is a limit system being a controlled diffusion process.
- Inspired by their work, we aim to develop near-optimal policies in this paper in an infinite horizon.

Some Assumptions

- Suppose $\xi(t)$ is a pure jump Markov-Feller process taking values in a compact metric space $\mathscr{S}$.
- Suppose its generator is given by

$$Q\phi(w) = q(w)\int_{\mathscr{S}} \Lambda(w, d\tilde{w})\phi(\tilde{w}) - q(w)\phi(w)$$

  where $q(\cdot)$ is continuous on $\mathscr{S}$ and $\Lambda(w,\cdot)$ is a probability measure on $\mathscr{S}$ for each w.
- Suppose that $\xi(t)$ is uniformly geometric ergodic with invariant measure $\overline{P}(\cdot)$. Let $\chi(w,\cdot) = \int_0^\infty \left[P(t,w,\cdot) - \overline{P}(\cdot)\right] dt$. Suppose that

$$r_i(\cdot) \text{ is bounded in } \mathscr{S}, \text{ and } \int_{\mathscr{S}} r_i(w)\overline{P}(dw) = 0,\ i = 1,2. \tag{5.3}$$

- Let $A = (a_{ij})_{2\times 2}$ with

$$a_{ij} := \int_{\mathscr{S}} \int_{\mathscr{S}} \chi(w, d\tilde{w})\overline{P}(dw)\Big[r_i(w)r_j(\tilde{w}) + r_j(w)r_i(\tilde{w})\Big].$$

- We suppose that A is positive definite with square root $(\sigma_{ij})_{2\times 2}$. It is well-known that in each finite interval of time, (5.1) can be approximated by

$$\begin{cases} dX(t) = & X(t)\big[\overline{a}_1 - b_1X(t) - c_1Y(t)\big]dt \\ & +X(t)(\sigma_{11}dW_1(t) + \sigma_{12}dW_2(t)) \\ dY(t) = & Y(t)\big[\overline{a}_2 - h(Y(t))u(t) - b_2Y(t) + c_2X(t)\big] \\ & dt + Y(t)(\sigma_{12}dW_1(t) + \sigma_{22}dW_2(t)), \end{cases} \qquad (5.4)$$

where $\overline{a}_1 = a_1 + \dfrac{a_{11}}{2} = a_1 + \dfrac{\sigma_{11}^2 + \sigma_{12}^2}{2}$,
$\overline{a}_2 = -a_2 + \dfrac{a_{22}}{2} = -a_2 + \dfrac{\sigma_{22}^2 + \sigma_{12}^2}{2}$, W_1, W_2 are two independent Brownian motions.

- By an ergodicity argument, it can be shown that if $-a_2 + c_2 \dfrac{a_1}{b_1} < 0$ then for any admissible control $u(t)$, $Y^\varepsilon(t)$ tends to 0 with probability 1 for any $\varepsilon > 0$, which implies

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi\Big(h(Y^\varepsilon(t))Y^\varepsilon(t)u(t)\Big)dt = 0 \text{ a.s.}$$

- Thus, to avoid the trivial limit, we assume that

$$-a_2 + c_2 \frac{a_1}{b_1} > 0. \tag{5.5}$$

Main Difficulty and Ideas to Solve

From the work of Kushner and Ruggaldier, we proceed to verify the following conditions to prove the desired result.

- (C1) There is an $\varepsilon_0 > 0$ such that $\{Z^\varepsilon(u,t), u \in PM^\varepsilon, 0 \le t < \infty, \varepsilon \le \varepsilon_0\}$ is $\mathbb{P}_{z,w}$-tight in $\mathbb{R}_+^{2,\circ}$ for each $(z,w) \in \mathbb{R}_+^{2,\circ} \times \mathscr{S}$.
- (C2) There is a δ-optimal Markov control $u(z)$ that is locally Lipschitz in z for any $\delta > 0$.

Once the conditions (C1) and (C2) are verified, we can use near-optimal strategies for (5.4) to construct near-optimal strategies for the harvested system perturbed by wideband noise.

- To achieve our goal of obtaining near optimality, one of our main tasks is to prove the uniform tightness of the ε-controlled system (5.1).
- Because of the negative quadratic terms in the drift, for the Lyapunov function $\widehat{V}_1(x,y) := 1 + c_2 x + c_1 y$, we can obtain that

$$\mathscr{L}_u \widehat{V}_1(x,y) \le \widehat{c}_1 - \widehat{c}_2 \widehat{V}_1^2(x,y)$$

for some $\widehat{c}_1 > 0, \widehat{c}_2 > 0$.
- By the perturbed Lyapunov function methods, we can construct a perturbation $\widehat{V}_1^\varepsilon$ satisfying

$$\mathscr{L}_u^\varepsilon \widehat{V}_1^\varepsilon \le \widehat{c}_3 - \widehat{c}_4 \widehat{V}_1^\varepsilon(x,y), (x,y) \in \mathbb{R}_+^{2,\circ} \tag{5.6}$$

for some $\widehat{c}_3 > 0$, $\widehat{c}_4 > 0$, and ε sufficiently small.
- This proves the uniform tightness of $\{Z^\varepsilon(t), u \in PM^\varepsilon, 0 \le t < \infty, \varepsilon \le \varepsilon_0\}$ in $\mathbb{R}_+^2$.

- The main issue is to investigate the behaviors of $Z^{\varepsilon}(t)$ near the boundary in order to obtain the uniform tightness in $\mathbb{R}_{+}^{2,\circ}$.
- While it seems practically impossible to find a Lyapunov function $\widehat{V}_2(x,y)$ satisfying $\widehat{V}_2(x,y) > 0, (x,y) \in \mathbb{R}_{+}^{2,\circ}$, $\widehat{V}_2(x,y) \to \infty$ as $(x,y) \to \partial\mathbb{R}_{+}^{2}$ and that

$$\mathscr{L}_u \widehat{V}_2(x,y) \le \widehat{c}_5 - \widehat{c}_6 \widehat{V}_2(x,y),$$

  investigating the long-term behavior of $Z(t)$ when it is close to the boundary as well as the invariant measures on the boundary provides an intuitive idea to tackle the problem.

- With a Markov control $m_t = v(Z(t)$, (5.4) has two ergodic invariant measures on the boundary $\partial\mathbb{R}^2_+$ which are the Dirac measure δ^* concentrated on $(0,0)$ and μ^* on $(0,\infty)\times\{0\}$.
- The growth rate of $X(t)$ when $Z(t)$ is close to $(0,0)$ is approximated by $a_1 > 0$. Likewise, the growth rate of $Y(t)$ when $Z(t)$ is close to $(0,\infty)\times\{0\}$ is approximated by $\int_0^t (\overline{a}_2 - h(y)v(x,y) - b_2 y + c_2 x)\mu^*(dx,dy) = -a_2 + \frac{c_2 a_1}{b_1} > 0.$
- This indicates that both ergodic measures on the boundary are “repeller”.

- As a result, by the idea of Lyapunov exponents and the use of log-Laplace approach we have the estimate that

$$\mathbb{E}_{x,y}\widehat{V}_2(X,(T)Y(T)) \le \kappa\widehat{V}_2(x,y)$$

where T is sufficiently large, $\kappa \in (0,1)$ and (x,y) is in a given bounded set and $\widehat{V}_2(x,y) = \dfrac{1}{x^{p_1}y^{p_2}}$ for suitable $p_1 > 0, p_2 > 0$.
- Then, we can derive

$$\mathbb{E}_{x,y}\widehat{V}_2^{\varepsilon}(X,(T)Y(T)) \le \kappa\widehat{V}_2^{\varepsilon}(x,y) \tag{5.7}$$

when (x,y) is in a bounded set and $\widehat{V}_2^{\varepsilon}(x,y)$ is a perturbation of $\widehat{V}_2$.
- Having that, we can get the uniform tightness in $\mathbb{R}_+^{2,\circ}$.

Near Optimality

Theorem 5.1

For any $\delta > 0$, there exists a locally Lipschitz Markov control u^δ such that for sufficiently small $\varepsilon > 0$, we have

$$\rho^* - 2\delta \le J^\varepsilon(u^\delta) := \liminf_{T\to\infty} \frac{1}{T}\int_0^T \Phi\Big(h(Y^\varepsilon(t)Y^\varepsilon(t))u^\delta(t)\Big)dt \le \rho^* + 2\delta \ \text{a.s.}$$

and that $\rho^* - \delta \le \mathfrak{J}^\varepsilon \le \rho^* + \delta$ a.s. which implies

$$J^\varepsilon(u^\delta) \ge \mathfrak{J}^\varepsilon - 3\delta \ \text{a.s.}$$

Here ρ^* is the optimal yield of (5.4) *and $\mathfrak{J}^\varepsilon$ is the optimal yield of* (5.1).

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$$v^*(x,y) = \begin{cases} 0 & \text{if } y < y^*(x) \\ M & \text{otherwise} \end{cases}$$

- We want to consider more general models and harvested ecosystems with constraints.

Further Remarks for the harvested model under wideband noise

The main result, Theorem 5.1 still holds true if the following generalizations are made.

(a) The coefficients $a_i, b_i, c_i, i = 1, 2$ depend on the state of $\xi^\varepsilon(t)$.

(b) The wideband noise in (5.1), which is linear in the current setup, can be replaced by nonlinear terms.

(c) The assumption on $\xi(t)$ in Section 2 can be reduced to the condition that $\xi(t)$ a stationary zero mean process, which is either (i) strongly mixing, right continuous and bounded, with the mixing rate function $\phi(\cdot)$ satisfying $\int_0^\infty \phi^{1/2}(s)ds < \infty$, or (ii) stationary Gauss-Markov with an integrable correlation function

Thank you